Local Structure of Compactified Jacobians

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Fine Print

This work is joint with:

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Alternative approaches to results are given by:

- the Chai-Faltings-Mumford theory of Uniformization (Aleexev and Nakamura);
- the theory of the Presentation Scheme (Oda and Seshadri).

We will work over the complex numbers $k := \mathbb{C}$.
(But ask if you are curious about a more general k!)

The Jacobian variety

To a non-singular curve X of genus g, one can associated the Jacobian:

$$\begin{aligned} \mathrm{J}_X^d &= \text{the Jacobian variety,} \\ &= \text{the moduli space of (degree } d\text{) line bundles} \\ &= \text{a complex torus.} \end{aligned}$$

The Torelli Theorem

The Jacobian is a basic tool for studying X.

Theorem (Torelli)

If

$$\mathrm{J}^0(X) \cong \mathrm{J}^0(Y) \text{ (with polarization)},$$

then

$$X \cong Y.$$

Question

Question (Mayer and Mumford, 1964)

1. *Is there an analogue when X is nodal?*
2. *If yes, do they fit into a family over $\overline{\mathrm{M}}_g$?*

Will focus on on Question 1 for a specific curve.

Only generalize J_X^d for $d = 0$.

Write $\overline{\mathrm{J}}_X^0$ for analogue of J_X^0.

A Genus 3 Curve

Draw Picture of Genus 3 **curve whose dual graph is** 2 **vertices joined by** 4 **edges.**

Attempt One

Form the moduli space of degree 0 line bundles!

Fails! This does not give a well-behaved scheme.

In genus 3 example, have new invariant:

$$\text{bidegree of } L = (\deg L|_{X_1}, \deg L|_{X_2})$$

Attempt One: Moduli Space

Draw infinite collection of copies of $\oplus_{i=1}^{3}\mathbb{C}^*$ indexed by possible bidegrees.

Attempt One: Problems

The problems are:

- the moduli space is NOT of finite type;
- the moduli space is NOT universally closed;
- more problems in a family (NOT separated).

Attempt Two

Construct as a GIT quotient of a Quot scheme!

Assume $d >> 0$. Form

$$U = \{(L; s_1, \ldots, s_r) : s_1, \ldots, s_r \in H^0(L) \text{ basis}\},$$

and the natural compactification

$$\mathrm{Quot}(X, \mathcal{O}^r) \supset U.$$

Attempt Two: Construction

Have (linearized) action of SL_r given by change of basis.

Form GIT quotient

$$\bar{\mathrm{J}}_X^d := \mathrm{Quot}(X, \mathcal{O}^r)//\,\mathrm{SL}_r\,.$$

Attempt Two: Theorem

Hard part: How to interpret points of $\overline{J}_X^d$?

Theorem (Caporaso-Pandharipande-Simpson)

The scheme $\overline{J}_X^d$ is a coarse moduli space of slope-stable rank 1, torsion-free sheaves.

Lots of generalizations. I know 10(!) other papers on this subject.

Attempt Two: Example

In genus 3 example, $\overline{J}^0(X)$ has 3 irreducible components.

Parameterizes line bundles of bidegree

$$(-2,2),(-1,1),(0,0),(1,-1),(2,-2).$$

and their degenerations.

Question

What is the local structure of $\overline{J}^0_X$ *at* $I := f_*(\mathcal{O}_{X'}(-2,-2))$*? How many local components?*

Proposition (Example)

There is isomorphism

$$\text{the completed local ring of } \bar{J}^0_X \text{ at } I \cong R^H,$$

where

$$\begin{aligned} H :=& \operatorname{Aut}(I)/\{scalars\} \\ =& (\mathbb{C}^* \times \mathbb{C}^*)/\mathbb{C}^* \end{aligned}$$

acting on

$$\begin{aligned} R :=& \hat{\bigotimes}_{i=1}^{4} \mathbb{C}[[u_i, v_i]]/(u_i v_i) \\ =& \text{the miniversal deformation ring for } I. \end{aligned}$$

Local Structure: Proposition

Proposition

The group action is

$$u_i \xmapsto{(a,b)} ab^{-1}u_i,$$
$$v_i \xmapsto{(a,b)} ba^{-1}v_i.$$

Proof.

There are three inputs:

- Luna's Slice Theorem shows an action exists;
- Rim's Theorem shows action is unique;
- compute using deformation theory.

□

Local Structure: Main Theorem

Theorem (C.-M., K., V.)

Let I be a polystable rank 1, torsion-free sheaf on a nodal curve X that fails to be locally free at a set $\Sigma \subset X$. Then the completed local ring of $\overline{\mathrm{J}}^d(X)$ at I is isomorphic to a power series ring over the completed cographic ring of $\Gamma_X(\Sigma)$.

Local Structure: Example

In the example, the theorem states that the ring is generated by

$$
\begin{array}{ll}
x_1y_2, & x_1y_3, \\
x_1y_4, & x_2y_3, \\
x_2y_4, & x_3y_4, \\
y_1x_2, & y_1x_3, \\
y_1x_4, & y_2x_3, \\
y_2x_4, & y_3x_4,
\end{array}
$$

which correspond to oriented cycles.

Local Structure: Local Components

How many local components in the genus 3 example? The answer is:

$$14 = 8 + 6.$$

Which correspond to totally cyclic orientations.

End

Thank you!