# Local Structure of Compactified Jacobians

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This work is joint with:

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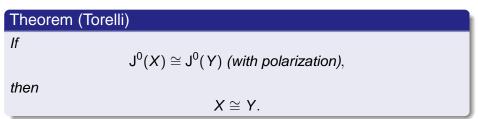
Alternative approaches to results are given by:

- the Chai-Faltings-Mumford theory of Uniformization (Aleexev and Nakamura);
- the theory of the Presentation Scheme (Oda and Seshadri).

We will work over the complex numbers  $k := \mathbb{C}$ . (But ask if you are curious about a more general k!) To a non-singular curve X of genus g, one can associated the Jacobian:

 $J_X^d$  =the Jacobian variety, =the moduli space of (degree *d*) line bundles =a complex torus.

## The Jacobian is a basic tool for studying X.



## Question (Mayer and Mumford, 1964)

- Is there an analogue when X is nodal?
- 2 If yes, do they fit into a family over  $\overline{M}_g$ ?

Will focus on on Question 1 for a specific curve.

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Only generalize J_X^d for d = 0.
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Write  $\overline{J}_X^0$  for analogue of  $J_X^0$ .

# Draw Picture of Genus 3 curve whose dual graph is 2 vertices joined by 4 edges.

Form the moduli space of degree 0 line bundles!

Fails! This does not give a well-behaved scheme.

In genus 3 example, have new invariant:

bidegree of  $L = (\deg L|_{X_1}, \deg L|_{X_2})$ 

# Draw infinite collection of copies of $\oplus_{i=1}^3 \mathbb{C}^*$ indexed by possible bidegrees.

The problems are:

- the moduli space is NOT of finite type;
- the moduli space is NOT universally closed;
- more problems in a family (NOT separated).

Construct as a GIT quotient of a Quot scheme!

Assume d >> 0. Form

$$U = \{(L; s_1, \ldots, s_r) : s_1, \ldots, s_r \in H^0(L) \text{ basis}\},\$$

and the natural compactification

 $\operatorname{Quot}(X, \mathcal{O}^r) \supset U.$ 

Have (linearized) action of  $SL_r$  given by change of basis.

Form GIT quotient

$$\overline{\mathsf{J}}^d_X := \operatorname{\mathsf{Quot}}(X, \mathcal{O}^r) / / \operatorname{\mathsf{SL}}_r.$$

Hard part: How to interpret points of  $\overline{J}_{X}^{d}$ ?

## Theorem (Caporaso-Pandharipande-Simpson)

The scheme  $\overline{J}_X^d$  is a coarse moduli space of slope-stable rank 1, torsion-free sheaves.

Lots of generalizations. I know 10(!) other papers on this subject.

In genus 3 example,  $\overline{J}^{0}(X)$  has 3 irreducible components.

Parameterizes line bundles of bidegree

$$(-2, 2), (-1, 1), (0, 0), (1, -1), (2, -2).$$

and their degenerations.

### Question

What is the local structure of  $\overline{J}_X^0$  at  $I := f_*(\mathcal{O}_{X'}(-2, -2))$ ? How many local components?

# Local Structure

## Proposition (Example)

There is isomorphism

the completed local ring of 
$$\overline{\mathsf{J}}_X^0$$
 at  $\mathsf{I}\cong\mathsf{R}^{\mathsf{H}},$ 

where

 $\begin{aligned} H &:= \operatorname{Aut}(I) / \{ \text{scalars} \} \\ &= (\mathbb{C}^* \times \mathbb{C}^*) / \mathbb{C}^* \end{aligned}$ 

acting on

 $R := \hat{\bigotimes}_{i=1}^{4} \mathbb{C}[[u_i, v_i]] / (u_i v_i)$ = the miniversal deformation ring for *I*.

# Local Structure: Proposition

# Proposition

#### The group action is

$$u_i \stackrel{(a,b)}{\longmapsto} ab^{-1}u_i,$$
  
 $v_i \stackrel{(a,b)}{\longmapsto} ba^{-1}v_j.$ 

### Proof.

There are three inputs:

- Luna's Slice Theorem shows an action exists;
- Rim's Theorem shows action is unique;
- compute using deformation theory.

## Theorem (C.-M., K., V.)

Let I be a polystable rank 1, torsion-free sheaf on a nodal curve X that fails to be locally free at a set  $\Sigma \subset X$ . Then the completed local ring of  $\overline{J}^{d}(X)$  at I is isomorphic to a power series ring over the completed cographic ring of  $\Gamma_X(\Sigma)$ .

#### In the example, the theorem states that the ring is generated by

<i>x</i> <sub>1</sub> <i>y</i> <sub>2</sub> ,	<i>x</i> <sub>1</sub> <i>y</i> <sub>3</sub> ,
<i>x</i> <sub>1</sub> <i>y</i> <sub>4</sub> ,	<i>x</i> <sub>2</sub> <i>y</i> <sub>3</sub> ,
<i>x</i> <sub>2</sub> <i>y</i> <sub>4</sub> ,	<i>x</i> <sub>3</sub> <i>y</i> <sub>4</sub> ,
<i>y</i> <sub>1</sub> <i>x</i> <sub>2</sub> ,	<i>y</i> <sub>1</sub> <i>x</i> <sub>3</sub> ,
<i>y</i> <sub>1</sub> <i>x</i> <sub>4</sub> ,	<i>y</i> <sub>2</sub> <i>x</i> <sub>3</sub> ,
<i>y</i> <sub>2</sub> <i>x</i> <sub>4</sub> ,	<i>y</i> <sub>3</sub> <i>x</i> <sub>4</sub> ,

which correspond to oriented cycles.

How many local components in the genus 3 example? The answer is:

14 = 8 + 6.

Which correspond to totally cyclic orientations.

Thank you!